

Exact solutions of Schrödinger equation for PT-/non-PT-symmetric and non-Hermitian Exponential Type Potentials with the position-dependent effective mass

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Abstract

Exact solutions of Schrödinger equation for PT-/non-PT-symmetric and non-Hermitian Morse and Pöschl-Teller potentials are obtained with the position-dependent effective mass by applying a point canonical transformation method. Three kinds of mass distributions are used in order to construct exactly solvable target potentials and obtain energy spectrum and corresponding wave functions.

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1 Introduction

In the past few years, theoretical researches on great variety of non-Hermitian Hamiltonians have received an important increase. Because many of these systems are invariant under combined parity and time reversal (PT) transformation which lead to either real (in case of broken PT symmetry) or pairs of complex conjugate energy eigenvalues (in case of spontaneously broken PT symmetry) [1,2]. This property of energy eigenvalues in non-Hermitian PT invariant systems can be related to the pseudo-hermiticity [3] or anti-unitary symmetry [4,5] of the corresponding Hamiltonians. In ref. [6] it was proposed a new class of non-Hermitian Hamiltonians with real spectra which are obtained using pseudo-symmetry. Moreover, completeness and orthonormality conditions for eigenstates of such potentials are proposed [7]. In the study of PT-invariant potentials various techniques have been applied to a great variety of quantum mechanical fields as variational methods, numerical approaches, Fourier analysis, semi-classical estimates, quantum field theory and Lie group theoretical approaches [7-16]. In additional, PT-symmetric and non-PT symmetric and also non-Hermitian potential cases such as oscillator type potentials and a variety of potentials within the framework of SUSYQM [17-21], exponential type screened potentials [22], quasi/conditionally exactly solvable ones [23], PT-symmetric and non-PT symmetric and also non-Hermitian potential cases within the framework of SUSYQM via Hamiltonian Hierarchy Method [24] and some others are studied [25-27].

On the other hand, there has been respected interest in a position-dependent mass, which is generally written as (PDM) $M(r) = m_0 m(r)$, problems associated with a quantum mechanical particle forms an effective model for the study of many physical problems [28-39] due to considerable applications in condensed matter physics and material science. The model applied to wide variety of physical systems such as quantum dots [40], liquid crystals [41], kinetics of evolution of microstructures and atomic displacements in the string [42], He cluster [43] semiconductor heterostructures [44] and nuclei [45]. Generally, those works are concentrated in obtaining the energy eigenvalues and the potential function for the given quantum system with the PDM. In the mapping of nonconstant mass Schrödinger equation, point canonical transformations (PCTs) are employed [46-49]. During the process, it is needed to transform non-constant mass, which is known as "effective mass" characterizes

the curvature of the dispersion relation, to a constant one so that the latter equation can be solved. Hence, energy spectra and corresponding wave functions of the target problem are produced easily. Various potentials, which satisfy the concept of exactly solvability, such as oscillator, Coulomb, Morse [50], hard-core potential [51], trigonometric type [52] and conditionally exactly solvable potentials [53] as well as the Scarf and Rosen-Morse type [54] ones including the PT-symmetry are considered for the construction of exact solution via PCT. The aim of this work is to apply PCT to the exact solutions of the nonconstant mass Schrödinger equation for Pöschl-Teller and Morse potentials which are complex and/or PT/non-PT symmetric, non-Hermitian and the exponential type systems.

The contents of the present paper is as follows: In section II, it is shown how to construct effective mass Schrödinger equation by using PCT method. In section III, IV and V, using three different type mass distributions, PCT method is applied to general Morse and non-Hermitian, PT/Non-PT symmetric Morse potentials. In section VI, VII and VIII, the general form of Pöschl-Teller potential and non-Hermitian, PT/Non-PT symmetric Pöschl-Teller potentials are studied by using PCT method within three different mass functions in order to construct the target problem including energy eigenvalues and corresponding wavefunctions within PT symmetry.

2 Effective Mass Schrödinger equation

As is well known, the general form of one dimensional time independent position-dependent mass Schrödinger equation (PDMSE) gives rise to

$$-\frac{1}{2} \left[\nabla_x \frac{1}{M(x)} \nabla_x \right] \psi(x) - [E - V(x)] \psi(x) = 0, \quad (1)$$

where $M(x) = m_0 m(x)$. So the Eq. (1) reads

$$\psi''(x) - \left(\frac{m'}{m} \right) \psi'(x) + 2m [E - V(x)] \psi(x) = 0, \quad (2)$$

where $\hbar = 1$ and m_0 is a constant. The one dimensional Schrödinger equation with a constant mass is

$$\Phi''(y) + 2[\varepsilon - V(y)]\Phi(y) = 0. \quad (3)$$

A transformation is defined as $y \rightarrow x$ and for a mapping $y = f(x)$, we rewrite the wave functions in the form

$$\Phi(y) = g(x)\psi(x) \quad (4)$$

The transformed Schrödinger equation reads

$$\psi''(x) + 2\left(\frac{g'}{g} - \frac{f''}{f'}\frac{g'}{g}\right)\psi'(x) + \left(\left(\frac{g''}{g} - \frac{f''}{f'}\frac{g'}{g}\right) + 2(f')^2[V(f(x) - \varepsilon)]\right)\psi(x) = 0. \quad (5)$$

Comparing Eqs. (2) and (5), we get the following identities

$$g(x) = \sqrt{\frac{f'(x)}{m(x)}} \quad (6)$$

and

$$V(x) - E = \frac{(f')^2}{m} [V(f(x) - \varepsilon)] - \frac{1}{2m}F(f, g) \quad (7)$$

where $F(f, g) = \left(\frac{g''}{g} - \frac{f''}{f'}\frac{g'}{g}\right)$. As it is seen from Eqs. (2) and (5), if we substitute $(f')^2 = m$ in Eq. (7), then the reference problem is transformed to the target problem including the energy spectra of the bound states, potential and wave function as

$$E_n = \varepsilon_n \quad (8)$$

$$V(x) = V(f(x)) - \frac{1}{8m} \left[\frac{m''}{m} - \frac{7}{4} \left(\frac{m'}{m} \right)^2 \right] \quad (9)$$

$$\psi(x) = [m(x)]^{1/4} \Phi_n(f(x)). \quad (10)$$

The PCT method can be applied to a problem which has an exact solution by using the procedure given below.

3 Generalized Morse Potential

Consider the Morse potential as the reference problem [19,22]

$$V(y) = V_1 e^{-2\alpha y} - V_2 e^{-\alpha y} \quad (11)$$

The energy eigenvalues and eigenfunctions of the our source potential are given as

$$\varepsilon_n = -\frac{\alpha^2}{4} \left[\frac{V_2}{\alpha \sqrt{V_1}} - (2n + 1) \right]^2 \quad (12)$$

$$\Phi_n(y) = C_n s^{2\epsilon} e^{-\gamma s} L_n^{4\epsilon}(2\gamma s) \quad (13)$$

where $s = \sqrt{V_1} e^{-\alpha y}$.

3.1 Asymptotically vanishing mass distribution

In this section, we use asymptotically vanishing type mass distribution as given below in order to get some target potentials providing us the exact solutions

$$m(x) = \frac{\alpha^2}{x^2 + q} \quad (14)$$

The mapping function becomes

$$y = f(x) = \int^x \sqrt{m(x)} dx = \alpha \ln \left(x + \sqrt{x^2 + q} \right) \quad (15)$$

and

$$x = \sinh_q \left(\frac{y}{\alpha} \right), \alpha \neq 0. \quad (16)$$

Using Eqs. (12-16), the new potential is obtained as

$$V(x) = V_1 \left(x + \sqrt{x^2 + q} \right)^{-2\alpha^2} - V_2 \left(x + \sqrt{x^2 + q} \right)^{-\alpha^2} - \frac{1}{8\alpha^2} \left(1 + \frac{q}{x^2 + q} \right) \quad (17)$$

Hence, the energy eigenvalues and corresponding wave functions for the general Morse potential are obtained as

$$E_n = \varepsilon_n \quad (18)$$

and

$$\psi_n(x) = C_n \frac{\sqrt{\alpha}}{(x^2 + q)^{1/4}} (f(x))^{2\epsilon} e^{-\gamma f(x)} L_n^{4\epsilon}(2\gamma f(x)). \quad (19)$$

where $\epsilon^2 = -\frac{E}{2\alpha^2}$, $\gamma = \frac{1}{\alpha^2}$.

3.2 Mass Distribution $m(x) = \frac{\alpha^2}{(b+x^2)^2}$

In the second example of mass distribution, the mapping function becomes

$$y = f(x) = \alpha \tan^{-1} \frac{x}{q} \quad (20)$$

and

$$x = \sqrt{q} \tan\left(\frac{y}{\alpha}\right) \quad (21)$$

The mapping function leads to the following target system having the same energy spectra

$$V(x) = V_1 e^{-2\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}} - V_2 e^{-\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}} - \frac{73x^2 - 16q}{32\alpha^2} \quad (22)$$

and

$$\psi_n(x) = \sqrt{\frac{q + x^2}{\alpha}} (f(x))^{2\epsilon} e^{-\gamma f(x)} L_n^{4\epsilon}(2\gamma f(x)). \quad (23)$$

3.3 Exponential Type Mass Distribution

If we consider the third type exponential mass function given as

$$m(x) = e^{-\alpha x} \quad (24)$$

$$y = f(x) = -\frac{2}{\alpha} e^{-\frac{\alpha x}{2}} \quad (25)$$

and

$$x = -\frac{2}{\alpha} \ln\left(-\frac{\alpha y}{2}\right) \quad (26)$$

The mapping yields to potential with the same energy spectra

$$V(x) = V_1 e^{4e^{-\alpha x/2}} - V_2 e^{2e^{-\alpha x/2}} + \frac{3\alpha^2 e^{-\alpha x}}{32}. \quad (27)$$

and corresponding wave function is

$$\psi_n(x) = (f(x))^{2\epsilon} e^{-\alpha x/4 - \gamma f(x)} L_n^{4\epsilon}(2\gamma f(x)) \quad (28)$$

4 Non-PT symmetric and non-Hermitian Morse Potential

In the equation (11), if the potential parameters are defined as $V_1 = (A+iB)^2$, $V_2 = (2C+1)(A+iB)$ and $\alpha = 1$, then the potential becomes

$$V(y) = (A + iB)^2 e^{-2y} - (2C + 1)(A + iB)e^{-y} \quad (29)$$

where A , B and C are arbitrary real parameters and $i = \sqrt{-1}$. Similarly, the energy eigenvalues for the reference potential is given as [19,22]

$$\varepsilon_n = -(n - C)^2 \quad (30)$$

4.1 Asymptotically vanishing mass distribution

Following the same procedure as in above, we get the target system

$$E_n = \varepsilon_n \quad (31)$$

$$V(x) = (A + iB)^2 \left(x + \sqrt{x^2 + q} \right)^{-2\alpha} - (2C + 1)(A + iB) \left(x + \sqrt{x^2 + q} \right)^{-\alpha} - \frac{1}{8\alpha^2} \left(1 + \frac{q}{x^2 + q} \right). \quad (32)$$

4.2 Mass Distribution $m(x) = \frac{\alpha^2}{(b+x^2)^2}$

Following the same procedure, we obtain the target potential with same energy spectra and

$$V(x) = (A + iB)^2 e^{-2\alpha \tan^{-1} \frac{x}{\sqrt{q}}} - (2C + 1)(A + iB) e^{-\alpha \tan^{-1} \frac{x}{\sqrt{q}}} - \frac{73x^2 - 16q}{32\alpha^2} \quad (33)$$

4.3 Exponential Type Mass Distribution

We obtain the target potential with same energy spectrum for exponential type mapping function as

$$V(x) = (A + iB)^2 e^{\frac{4e^{-\frac{\alpha x}{2}}}{\alpha}} - (2C + 1)(A + iB)e^{\frac{2e^{-\frac{\alpha x}{2}}}{\alpha}} + \frac{3\alpha^2 e^{-\alpha x}}{32}. \quad (34)$$

5 PT symmetric and non-Hermitian Morse Potential

When $\alpha = i\alpha$ and V_1, V_2 are real, the Morse potential becomes

$$V(y) = V_1 e^{-2i\alpha y} - V_2 e^{-i\alpha y} \quad (35)$$

The energy eigenvalues are given for this potential as [19,22]

$$\varepsilon_n = \alpha^4 \left[\left(n + \frac{1}{2} \right) + \frac{V_2}{2\alpha\sqrt{|-V_1|}} \right]^2 \quad (36)$$

If we take the parameters of Eq.(25) as $V_1 = -\omega^2$, $V_2 = D$ and $\alpha = 2$ then, corresponding energy eigenvalues for any n-th state are,

$$\varepsilon_n = \left(2n + 1 + \frac{D}{2\omega} \right)^2 \quad (37)$$

which is studied by Znojil and Bagchi and Quesne. [10-11,19].

5.1 Asymptotically vanishing mass distribution

Thus, the target system with asymptotically vanishing mass distribution are given as

$$E_n = \varepsilon_n \quad (38)$$

$$V(x) = V_1 \left(x + \sqrt{x^2 + q} \right)^{-2i\alpha^2} - V_2 \left(x + \sqrt{x^2 + q} \right)^{-i\alpha^2} - \frac{1}{8\alpha^2} \left(1 + \frac{q}{x^2 + q} \right). \quad (39)$$

5.2 Mass Distribution $m(x) = \frac{\alpha^2}{(b+x^2)^2}$

In the PT symmetric and non-Hermitian case, new potential is given by

$$V(x) = V_1 \sqrt{\frac{1 - 2\alpha^2 \frac{x}{\sqrt{q}}}{1 + 2\alpha^2 \frac{x}{\sqrt{q}}}} - V_2 \sqrt{\frac{1 - \alpha^2 \frac{x}{\sqrt{q}}}{1 + \alpha^2 \frac{x}{\sqrt{q}}}} - \frac{73x^2 - 16q}{32\alpha^2}. \quad (40)$$

5.3 Exponential Type Mass Distribution

The new potential with the same energy spectra is

$$V(x) = V_1 e^{4ie^{-\alpha x/2}} - V_2 e^{2ie^{-\alpha x/2}} + \frac{3\alpha^2 e^{-\alpha x}}{32}. \quad (41)$$

6 Pöschl-Teller Potential

The general form of the Pöschl-Teller potential is [19,22]

$$V(y) = -4V_0 \frac{e^{-2\alpha y}}{(1 + qe^{-2\alpha y})^2} \quad (42)$$

Its energy spectra and corresponding wavefunctions are

$$\varepsilon_n = -\frac{\alpha^2}{4} \left(-(2n+1) + \sqrt{1 + \frac{8V_0}{q\alpha^2}} \right)^2 \quad (43)$$

$$\psi_n(y) = s^{-\epsilon} (1-s)^{\nu/2} P_n^{(2\epsilon, \nu-1)}(1-2qs). \quad (44)$$

where $s = -e^{-2\alpha y}$, $P_n^{-\nu_2-\frac{1}{2}, \nu_2-\frac{1}{2}}(y)$ stands for Jacobi polynomials and $\nu_1 = \sqrt{1 + \frac{8V_0}{q\alpha^2}}$, $\nu_2 = \sqrt{\frac{8V_0}{q\alpha^2}}$.

6.1 Asymptotically vanishing mass distribution

The target system is obtained as with the mass function

$$E_n = \varepsilon_n \quad (45)$$

$$V(x) = -4V_0 \frac{(x + \sqrt{x^2 + q})^{-2\alpha^2}}{\left[1 + q(x + \sqrt{x^2 + q})^{-2\alpha^2}\right]^2} - \frac{1}{8\alpha^2} \left(1 + \frac{q}{x^2 + q}\right) \quad (46)$$

$$\psi_n(x) = \frac{(x^2 + q)^{1/4}}{\sqrt{\alpha}} (f(x))^{-\epsilon} (1 - f(x))^{\nu/2} P_n^{(2\epsilon, \nu-1)}(1 - 2qf(x)) \quad (47)$$

6.2 Mass Distribution $m(x) = \frac{\alpha^2}{(q+x^2)^2}$

If we consider to obtain a target potential for the Pöschl-Teller Potential, it can be obtain as

$$V(x) = -4V_0 \frac{e^{-2\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}}}{\left(1 + qe^{-2\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}}\right)^2} - \frac{73x^2 - 16q}{32\alpha^2}. \quad (48)$$

$$\psi_n(x) = \frac{(x^2 + q)^{1/2}}{\sqrt{\alpha}} (f(x))^{-\epsilon} (1 - f(x))^{\nu/2} P_n^{(2\epsilon, \nu-1)}(1 - 2qf(x)) \quad (49)$$

6.3 Exponential Type Mass Distribution

With the exponential Type Mass Distribution, it can be obtained with the same energy spectra as

$$V(x) = -4V_0 \frac{e^{4e^{-\alpha x/2}}}{[1 + qe^{4e^{-\alpha x/2}}]^2} + \frac{3\alpha^2 e^{-\alpha x}}{32}. \quad (50)$$

and

$$\psi_n(x) = e^{\alpha x/4} (f(x))^{-\epsilon} (1 - f(x))^{\nu/2} P_n^{(2\epsilon, \nu-1)}(1 - 2qf(x)) \quad (51)$$

7 Non-PT symmetric and non-Hermitian Pöschl-Teller cases

In this case, V_0 and q are complex parameters $V_0 = V_{0R} + iV_{0I}$ and $q = q_R + iq_I$ but α is a real parameter. Although the potential is complex and the corresponding Hamiltonian is non-Hermitian and also non-PT symmetric, there may be real spectra if and only if $V_{0I}q_R = V_{0R}q_I$. When both parameters V_0 and q are taken pure imaginary, the potential turns out to be [19,22],

$$V(y) = -4V_0 \frac{2qe^{-4\alpha y} + i(1 - q^2e^{-4\alpha y})}{(1 + q^2e^{-4\alpha y})^2} \quad (52)$$

For simplicity, we use the notation V_0 and q instead of V_{0I} and q_I . In this case, the same energy eigenvalues are obtained as in the Eq.(30).

7.1 Asymptotically vanishing mass distribution

The new potential is

$$V(x) = -4V_0 \frac{2q \left(x + \sqrt{x^2 + q}\right)^{-4\alpha^2} + i \left(1 - q^2 \left(x + \sqrt{x^2 + q}\right)^{-4\alpha^2}\right)}{\left[1 + q^2 \left(x + \sqrt{x^2 + q}\right)^{-4\alpha^2}\right]^2} - \frac{1}{8\alpha^2} \left(1 + \frac{q}{x^2 + q}\right). \quad (53)$$

7.2 Mass Distribution $m(x) = \frac{\alpha^2}{(b+x^2)^2}$

The target potential for this case can be obtained as

$$V(x) = -4V_0 \frac{2qe^{-4\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}} + i \left(1 - q^2 e^{-4\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}}\right)}{\left(1 + q^2 e^{-4\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}}\right)^2} - \frac{73x^2 - 16q}{32\alpha^2}. \quad (54)$$

7.3 Exponential Type Mass Distribution

The target potential for this case can be obtained with same the energy eigenvalues as

$$V(x) = -4V_0 \frac{2qe^{8e^{-\alpha x/2}} + i(1 - q^2)e^{8e^{-\alpha x/2}}}{(1 + q^2e^{8e^{-\alpha x/2}})^2} + \frac{3\alpha^2 e^{-\alpha x}}{32}. \quad (55)$$

8 PT symmetric and non-Hermitian Pöschl-Teller cases

We choose the parameters V_0 and q are real and also $\alpha = i\alpha$. Then, the potential turns into

$$V(x) = -4V_0 \frac{(1+q^2)\cos 2\alpha x + 2q + i(q^2-1)\sin 2\alpha x}{(1+q^2)^2 + 4q\cos 2\alpha x((1+q\cos 2\alpha x + q^2))} \quad (56)$$

and corresponding energy eigenvalue is given as [19,22]

$$\varepsilon_n = -\frac{\alpha^2}{4} \left[2n + 1 + \sqrt{1 + \frac{16V_0}{\alpha^2}} \right]^2 \quad (57)$$

8.1 Asymptotically vanishing mass distribution

The new system is

$$E_n = \varepsilon_n \quad (58)$$

$$V(x) = -4V_0 \frac{\left[q \left(x + \sqrt{q+x^2} \right)^{i\alpha^2} - \left(x + \sqrt{q+x^2} \right)^{-i\alpha^2} \right]^2}{(1+q^2)^2 + 4q\cos \left[2\alpha^2 \ln \left(x + \sqrt{q+x^2} \right) \right] \left(1 + q\cos \left[2\alpha^2 \ln \left(x + \sqrt{q+x^2} \right) \right] + q^2 \right)} - \frac{1}{8\alpha^2} \left(1 + \frac{q}{x^2+q} \right) \quad (59)$$

8.2 Mass Distribution $m(x) = \frac{\alpha^2}{(b+x^2)^2}$

The new potential is given as

$$V(x) = -4V_0 \frac{(qe^{i\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}} + e^{-i\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}}})^2}{(1+q^2)^2 + 4q\cos(2\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}})(1 + q\cos 2\alpha^2 \tan^{-1} \frac{x}{\sqrt{q}} + q^2)} - \frac{73x^2 - 16q}{32\alpha^2}. \quad (60)$$

8.3 Exponential Type Mass Distribution

The target potential can be obtained with having the same energy spectra for this case as

$$V(x) = -4V_0 \frac{(qe^{-2ie^{-\alpha x/2}} + e^{2ie^{-\alpha x/2}})^2}{(1+q^2)^2 + 4q\cos(4e^{-\alpha x/2})(1 + q\cos(4e^{-\alpha x/2}) + q^2)} + \frac{3\alpha^2 e^{-\alpha x}}{32}. \quad (61)$$

9 Conclusions

In this article we have explored the PCT approach to a class of exponential type PT/non-PT symmetric and nonhermitian Hamiltonians such as Morse and Pöschl-Teller potentials with some spatially dependent effective masses. For each type of potentials we have obtained a set of exactly solvable target potentials by using three position dependent mass distributions. It is pointed out that the importance of the mapping function which aids to construct of closed forms of the energy spectrum and corresponding wavefunctions. Specially, the reference system with the source potential and the new system with the target potential have the same bound state energy eigenvalues. It was shown that the exact solvability depends not only on the form of the potential, but also on the spatial dependence on the mass within the PT symmetric framework.

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